ON TOPOLOGICAL PROPERTIES OF FAMILIES OF FINITE SETS

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ABSTRACT. We present results about the Cantor-Bendixson index of some subspaces of a uniform family $\mathcal{F}$ of finite subsets of natural numbers with respect to the lexicographic order topology. As a corollary of our results we get that for any ω-uniform family $\mathcal{F}$ the restriction $\mathcal{F} \upharpoonright M$ is homeomorphic to $\mathcal{F}$ iff M contains intervals of arbitrary length of consecutive integers. We show the connection of these results with a topological partition problem of uniform families.

1. INTRODUCTION

A partition problem for topological spaces is as follows: Given spaces X and Y and a partition of X into two pieces, is there a topological copy of Y inside one of the pieces? When the answer is positive, it is denoted by $X \to (Y)^1_2$ (see [4] for more information about this type of problems). We will be mainly interested in the case $X = Y$. A well studied case is when X is a countable ordinal endowed with its natural order topology. A result of Baumgartner [2] solves this partition problem for a countable ordinal space α. Namely, he showed that for a countable ordinal α, $\alpha \to (\alpha)^1_2$ iff α is of the form ω^{ω^β}.

Any countable ordinal is the order type of a uniform family $\mathcal{F}$ of finite subsets of natural numbers lexicographically ordered. A typical uniform family of order type ω^k is the collection of k-elements subsets of $\mathbb{N}$. Thus a partition of a countable ordinal space can be regarded as a partition of a uniform family endowed with the lexicographic order topology (the relevant definitions are given on section 2).

Families of finite sets has been the focus of Ramsey theory for a long time [3]. A well known result of Nash-Williams says that for any uniform family $\mathcal{F}$ on $\mathbb{N}$ and any subset $\mathcal{B}$ of $\mathcal{F}$ there is an infinite set $A \subseteq \mathbb{N}$ such that either $\mathcal{F} \upharpoonright A \subseteq \mathcal{B}$ or $\mathcal{F} \upharpoonright A \cap \mathcal{B} = \emptyset$ (see [3]) where $\mathcal{F} \upharpoonright A$ is the collection of elements of $\mathcal{F}$ that are subsets of A. This theorem solves the topological partition problem for $\mathcal{F}$, if the topological type of $\mathcal{F} \upharpoonright A$ and $\mathcal{F}$ are the same. This was the starting point for this research. We soon realized that $\mathcal{F} \upharpoonright A$ could be a discrete subspace of $\mathcal{F}$ and hence Baumgartner's theorem is not a corollary of the Nash-Williams's theorem. In fact, given a uniform family $\mathcal{F}$, there is $\mathcal{B} \subset \mathcal{F}$ such that $\mathcal{F} \upharpoonright A$ is a discrete subset of $\mathcal{F}$ for every set A homogeneous for the partition given by $\mathcal{B}$ (i.e. for any A satisfying the conclusion of Nash-Williams's theorem applied to $\mathcal{F}$ and $\mathcal{B}$) (see example 3.13). Nevertheless, it is natural to wonder about the topological type of $\mathcal{F} \upharpoonright A$. The objective of this paper is to present an analysis of the Cantor-Bendixson index of $\mathcal{F} \upharpoonright A$ as a subspace of a uniform family $\mathcal{F}$. Notice that $\mathcal{F} \upharpoonright A$ has the same order type of $\mathcal{F}$, but the topological type varies considerably depending on the set A. Hence the difficulty lies on the fact that we are using on $\mathcal{F} \upharpoonright A$ the subspace topology.

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To give an example of the results presented in this paper, we recall a typical ω-uniform family, the so called *Schreier barrier*:

$$\mathcal{S} = \{t \in \mathbb{N}^{[<\infty]} : \ |t| = min(t) + 1\}.$$

It is known that $\mathcal{S}$ is homeomorphic to ω^{ω}. We will show that $\mathcal{S} \restriction M$ contains a topological copy of $\mathcal{S}$ iff M contains intervals of consecutive integers of arbitrary length. Finally, we mention that besides the important role played by uniform families in Ramsey theory [3], they have also appeared in the theory of Banach spaces as tools for the construction of Tsirelson-like spaces [1].

The paper is organized as follows. In section 2 we introduce the terminology and some preliminary facts. In section 3 we study the Cantor-Bendixson derivatives of uniform families. In section 4 we introduce the type of sets M such that the restriction $\mathcal{F} \restriction M$ has the same Cantor-Bendixson index as $\mathcal{F}$. Finally, in section 5 we present the main results about when $\mathcal{F} \restriction M$ contains a topological copy of $\mathcal{F}$.

2. Preliminaries

We denote by $\mathbb{N}^{[<\infty]}$ the collection of all finite subsets of $\mathbb{N}$. If M is a set, $M^{[k]}$ denotes the collection of all k-elements subsets of M. By $M^{[\infty]}$ we denote the collection of all infinite subsets of M.

The lexicographic order $<_{lex}$ over $\mathbb{N}^{[<\infty]}$ is defined as follows: Given $s, t \in \mathbb{N}^{[<\infty]}$ we put $s <_{lex} t$ iff $min(s \triangle t) \in s$.

We write $s \sqsubseteq t$ when there is $n \in \mathbb{N}$ such that $s = t \cap \{0, 1, \cdots, n\}$ and we say that s is an initial segment of t. A collection $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is a *front* on M if satisfies the following conditions: (i) Every two elements of $\mathcal{F}$ are $\sqsubseteq$-incomparable. (ii) Every infinite subset N of M has an initial segment in $\mathcal{F}$.

Given $\mathcal{F} \subseteq \mathbb{N}^{[<\infty]}$ and $u \in \mathbb{N}^{[<\infty]}$, let

$$\mathcal{F}_u = \{s \in \mathbb{N}^{[<\infty]} : \ u \cup s \in \mathcal{F}, \ max(u) < min(s)\}.$$

For convenience, we set $max(\emptyset) = -1$; in particular, $\mathcal{F}_{\emptyset} = \mathcal{F}$.

For M an infinite subset of $\mathbb{N}$, let

$$\mathcal{F} \restriction M = \{s \in \mathcal{F} : \ s \subset M\}.$$

We put $M/k = \{n \in M : \ k < n\}$. If u is a finite set and $n = max(u)$, we put $M/u = M/n$. The notion of an α-uniform family on an infinite set M is defined by recursion.

(i) $\{\emptyset\}$ is the unique 0-uniform family on M.
(ii) $\mathcal{F} \subseteq \mathbb{N}^{[<\infty]}$ is said to be $(\alpha+1)$-uniform on M, if $\mathcal{F}_{\{n\}}$ is α-uniform on M/n for all $n \in M$.
(iii) If α is a limit ordinal, we say that $\mathcal{F}$ is α-uniform on M, if there is an increasing sequence $(\alpha_k)_{k \in M}$ converging to α such that $\mathcal{F}_{\{k\}}$ is α_k-uniform on M/k for all $k \in M$.

For $k \in \mathbb{N}$, $M^{[k]}$ is the unique k-uniform family on M. The following collection is an ω-uniform family on $\mathbb{N}$, called *Schreier barrier*:

$$\mathcal{S} = \{t \in \mathbb{N}^{[<\infty]} : \ |t| = min(t) + 1\}.$$

We say that $\mathcal{F}$ is uniform on M when it is α-uniform on M for some α. Notice that if $\mathcal{F}$ is uniform on M, then $\mathcal{F}_u$ es uniform on M/u.

The following result is well known [1].

Theorem 2.1. *Let $\mathcal{F}$ be an α-uniform family over M. Then $\mathcal{F}$ is a front over M and $\mathcal{F} \restriction N$ is α-uniform over N for all infinite $N \subseteq M$.*

Given a front $\mathcal{F}$ on a final segment S of $\mathbb{N}$. For $n \in S$, we denote by $t_n^{\mathcal{F}}$ the unique element of $\mathcal{F}$ verifying

$$t_n^{\mathcal{F}} \sqsubseteq \{n, n+1, n+2, \dots\}.$$

In the sequel, the sets $t_n^{\mathcal{F}_u}$ will be very useful. In particular, we remark that given a finite set $u \subset S$ and $n \in S/u$, there is a unique m such that

$$u \cup t_n^{\mathcal{F}_u} = u \cup \{n, n+1, \cdots, n+m\} \in \mathcal{F}.$$

Notice that if $s \in \mathcal{F}$ and $n = \min(s)$, then

$$t_n^{\mathcal{F}} \leq_{lex} s <_{lex} t_{n+1}^{\mathcal{F}}.$$

Given two families $\mathcal{F}$ and $\mathcal{G}$ of finite sets, define $\mathcal{F} \oplus \mathcal{G}$ as follows:

$$\mathcal{F} \oplus \mathcal{G} = \{s \cup t : \ s \in \mathcal{G}, \ t \in \mathcal{F} \text{ and } max(s) < min(t)\}.$$

If $\mathcal{F}$ is α-uniform and $\mathcal{G}$ is β-uniform, then $\mathcal{F} \oplus \mathcal{G}$ is $(\alpha + \beta)$-uniform. Notice that if $\mathcal{F}$ is a front over a final segment S of $\mathbb{N}$, then $t_n^{\mathcal{F}} = \min(\mathcal{F}_{\{n\}} \oplus \{\{n\}\}, <_{lex})$ for all $n \in S$.

The following result is well known (see for instance [1]).

Theorem 2.2. *Let $\mathcal{F}$ be an α-uniform family over a set M. Then $\mathcal{F}$ is lexicographically well ordered and its order type is ω^α.*

In what follows, we consider an uniform family $\mathcal{F}$ on $\mathbb{N}$ (or a final segment of $\mathbb{N}$) as topological space by giving $\mathcal{F}$ the order topology respect to the lexicographic order $<_{lex}$.

Now we recall some known facts about the Cantor-Bendixson derivative (CB derivative in short). Given a topological space X and $A \subseteq X$, we let A' be the set of all limit points $x \in A$. Recursively, $A^{(0)} = A$, $A^{(\alpha+1)}$ is $(A^{(\alpha)})'$ and for α a limit ordinal, $A^{(\alpha)}$ is $\bigcap_{\beta<\alpha} A^{(\beta)}$. The least α such that $A^{(\alpha)} = A^{(\alpha+1)}$ is called the CB index of A. It is well know that ω^α with the order topology has CB index equal to α.

An ordinal is said to be *indecomposable* if there are not $\beta, \gamma < \alpha$ such that $\alpha = \beta + \gamma$. It is known that α is indecomposable iff $\alpha = \omega^\beta$ for some β.

To get copies of uniform families we will use the following theorem which follows from the results in [2].

Theorem 2.3. *Let $\alpha < \omega_1$ be an indecomposable ordinal and $X \subseteq \omega^\alpha$. If $X^{(\gamma)} \neq \emptyset$ for all $\gamma < \alpha$, then X has a subspace homeomorphic to ω^α.*

3. CB derivatives of uniform families

In this section we study the behavior of the CB derivative on $\mathcal{F} \uparrow M$, for $M \in \mathbb{N}^{[\infty]}$, as a subspace of $\mathcal{F}$. In particular, we will characterize the limit points in $\mathcal{F} \restriction M$.

Lemma 3.1. *Let $\mathcal{F}$ be an α-uniform family on a final segment of $\mathbb{N}$ with $\alpha \geq \omega$ and $t \in \mathcal{F}$.*

(i) If $\alpha = \omega$, then $|t| \geq min(t) + 1$.
(ii) If $\alpha > \omega$, then $|t| > min(t) + 2$.

In particular, $|t| \geq 2$ for all t in an α-uniform family with $\alpha > 1$ and $min(t) \geq 1$.

Proof. Let $\mathcal{F}$ be an ω-uniform family and $t \in \mathcal{F}$. Let $n = min(t)$, then $t/n \in \mathcal{F}_{\{n\}}$ and $\mathcal{F}_{\{n\}}$ is k-uniform with $k \geq n$, therefore the size of t is at least $n+1$. The rest of the claim follows by induction on α. $\square$

Lemma 3.2. *Let $\mathcal{F}$ be an uniform family on a final segment of $\mathbb{N}$.*

(i) Suppose $(s_i)_i$ is a sequence in $\mathcal{F}$ such that $s_i \to s$ with $s \in \mathcal{F}$, then there exists $k \in \mathbb{N}$ such that $min(s) - 1 \leq min(s_i) \leq min(s)$ for all $i \geq k$. In particular, $s_i \leq_{lex} s$ for all $i \geq k$.

(ii) Suppose $(s_i)_i$ is a sequence in $\mathcal{F}$ of the form $s_i = u \cup \{p-1\} \cup v_i$ where $u \in \mathbb{N}^{[<\infty]}$, $p \geq 1$, $max(u) < p-1 < min(v_i)$ and $min(v_i) \to \infty$. Then there is $m \in \mathbb{N}$ such that

$$s_i \to u \cup \{p, p+1, \ldots, p+m\} = u \cup t_p^{\mathcal{F}_u}.$$

(iii) Suppose $(s_i)_i$ is a sequence in $\mathcal{F}$ such that $s_i \to s \in \mathcal{F}$ and $min(s_i) = min(s) - 1 = p-1$ for all i. Then $s = t_p^{\mathcal{F}}$ and $s_i = \{p-1\} \cup v_i$ for some v_i such that $p-1 < min(v_i)$ and $min(v_i) \to \infty$. Conversely, if $s_i \to_i t_p^{\mathcal{F}}$ and $s_i \neq t_p^{\mathcal{F}}$ for all i, then eventually $min(s_i) = p-1$.

(iv) Suppose $s_i \to s$ and $min(s_i) = min(s) = n$ for all i. Then $s_i/n \to s/n$.

(v) Suppose $s, s_i \in \mathcal{F}$ with $s \neq s_i$ for all i and $s_i \to s$. Then there are $u, v_i \in \mathbb{N}^{[<\infty]}$ and $p \in \mathbb{N}$ such that

$$s = u \cup t_p^{\mathcal{F}_u}$$

and eventually

$$s_i = u \cup \{p-1\} \cup v_i$$

where $max(u) < p-1 < min(v_i)$ and $min(v_i) \to \infty$.

Proof. (i) follows from the fact that $\mathcal{F}$ is a front and the topology of $\mathcal{F}$ is the order topology given by $<_{lex}$ which is a well-order on $\mathcal{F}$. In particular, convergence in $\mathcal{F}$ is from below.

To see (ii), let $s = u \cup t_p^{\mathcal{F}_u}$ and $w \in \mathcal{F}$ such that $w <_{lex} s$. It is clear that $s_i <_{lex} s$ for all i. We will show that eventually $w <_{lex} s_i$. The only interesting case is when $w = u \cup v$ with $max(u) < min(v)$. If $min(v) < p-1$, then clearly $w <_{lex} s_i$ for all i. Suppose then that $min(v) = p-1$. As $s_i \in \mathcal{F}$ and $\mathcal{F}$ is a $\sqsubseteq$-antichain, then $u \cup \{p-1\} \notin \mathcal{F}$ and thus $|v| \geq 2$. Therefore, $w <_{lex} s_i$ for all large enough i.

For (iii), notice that $s_i <_{lex} t_p^{\mathcal{F}} \leq_{lex} s$ for all i. Thus $s = t_p^{\mathcal{F}}$. Suppose m is such that $min(v_i) < m$. Since $\mathcal{F}$ is a front, pick $w_m \in \mathcal{F}$ such that $\{p-1, m\} \sqsubseteq w_m$. Then $s_i <_{lex} w_m <_{lex} s$. Hence there are only finitely many such v_i and thus $min(v_i) \to \infty$.

To see (v). By (i) we assume that $s_i \leq_{lex} s$ for all i. If $min(s_i) = min(s) - 1$ eventually, then apply (iii) to get the conclusion with $u = \emptyset$. If $min(s_i) = min(s) = n$, then by (iv), $s_i/n \to s/n$; by repeating this finitely many times we get that $s = u \cup w$, $s_i = u \cup w_i$ with $max(u) < w$, $max(u) < min(w_i)$, $min(w_i) = min(w) - 1$ and $w_i \to w$. Since $w, w_i \in \mathcal{F}_u$ and $\mathcal{F}_u$ is uniform on $\mathbb{N}/u$, then we apply (iii) to finish the proof.

$\square$

Remark 3.3. Let $\mathcal{F}$ be an uniform family on $\mathbb{N}$. If $\mathcal{F} \upharpoonright M$ is a closed subset of $\mathcal{F}$, then M is a final segment of $\mathbb{N}$. In fact, let $n \in M$, we show that $n+1 \in M$. Since $\mathcal{F} \upharpoonright M$ is a front on M, let $v_i \in \mathcal{F} \upharpoonright M$ such that $\{n, i\} \sqsubseteq v_i$ for $i \in M/n$. Then $v_i \to t_{n+1}^{\mathcal{F}}$, in particular $n+1 \in M$.

Using the previous results, we are ready to characterize limit points in uniform families.

Proposition 3.4. *Let $\mathcal{F}$ be an α-uniform family on a final segment S of $\mathbb{N}$ with $1 < \alpha < \omega_1$, $M \in S^{[\infty]}$ and $t \in \mathcal{F} \upharpoonright M$ with $min(t) > 1$. Then, $t \in (\mathcal{F} \upharpoonright M)'$ if, and only if, there is $u \in S^{[<\infty]}$ and $p \in \mathbb{N}$ such that*

$$t = u \cup \{p, p+1, \cdots, p+m\}$$

where $max(u) < p-1$, $p-1 \in M$ and $m \geq 1$. Notice that $t = u \cup t_p^{\mathcal{F}_u}$.

Proof. ($\Rightarrow$) Let $t \in (\mathcal{F} \upharpoonright M)'$, by lemma 3.2 we know that there is $u \in \mathbb{N}^{[<\infty]}$ and $p, m \in \mathbb{N}$ such that

$$t = u \cup \{p, p+1, \cdots, p+m\} = u \cup t_p^{\mathcal{F}_u}$$

and $max(u) < p-1$. Moreover, any sequence in $\mathcal{F} \upharpoonright M$ converging to t is eventually of the form $s_i = u \cup \{p-1\} \cup v_i$ where $max(u) < p-1 < min(v_i)$ and $min(v_i) \to \infty$. In particular, $p-1 \in M$. It remains only to show that $m \geq 1$. Since $\{p-1\} \cup v_i \in \mathcal{F}_u$, then $\mathcal{F}_u$ is not 1-uniform, thus by lemma 3.1, $t_p^{\mathcal{F}_u}$ has size at least 2, hence $m \geq 1$.

($\Leftarrow$) Reciprocally, suppose $t = u \cup t_p^{\mathcal{F}_u} \subseteq M$ for some $p \in M$ with $max(u) < p-1 \in M$. Notice that $\mathcal{F}_u \upharpoonright M$ is a β-uniform family on M/u for some $\beta < \alpha$. Since $t_p^{\mathcal{F}_u}$ has size at least 2, then $\beta \geq 2$. As $\mathcal{F}_u \upharpoonright M$ is a front on M/u, there is $w_i \in \mathcal{F}_u \upharpoonright M$ such that $\{p-1, i\} \sqsubseteq w_i$ for each $i \in M/(p-1)$. Then by lemma 3.2 we know that $u \cup w_i \to u \cup t_p^{\mathcal{F}_u}$. $\square$

Proposition 3.4 gives a tool to determine the topological type of a subspace $\mathcal{F} \upharpoonright M$. Also, it allows to construct subspaces $\mathcal{F} \upharpoonright M$ without copies of $\mathcal{F}$. The following example shows that $\mathcal{F} \upharpoonright M$ can be a discrete subspace of $\mathcal{F}$.

Examples 3.5. *For the following examples we shall consider the Schreier barrier $\mathcal{S}$ (defined in §2).*

- (i) *Let $M \in \mathbb{N}^{[\infty]}$ be the collection of even numbers. Since in M there are not consecutive numbers, then $\mathcal{S} \upharpoonright M$ is a discrete subspace of $\mathcal{S}$.*
- (ii) *Let $M = \{3k : k \in \mathbb{N}\}$ and $N = \mathbb{N} \backslash M$. In this case, N has consecutive numbers but $\mathcal{S} \upharpoonright N$ is also discrete, because $3q \notin N$ for all q.*

As we can see, given an uniform family $\mathcal{F}$ on $\mathbb{N}$, its restrictions $\mathcal{F} \upharpoonright M$ can change considerably its topological type. Nevertheless, for some sets M the restriction conserves the topological type of $\mathcal{F}$. The simplest example is when M is a final segment of $\mathbb{N}$, then $\mathcal{F} \upharpoonright M$ corresponds also to final segment of $\mathcal{F}$, therefore $\mathcal{F} \upharpoonright M$ is closed in $\mathcal{F}$ and the subspace topology of $\mathcal{F} \upharpoonright M$ is homeomorphic $\mathcal{F}$. But, as we shall show in following sections, there are also non trivial sets M such that $\mathcal{F} \upharpoonright M$ contains a topological copy of $\mathcal{F}$. To do this, we need to analyze the CB derivatives of an uniform family.

Using the definition of $\mathcal{F}_{\{n\}}$, $\oplus$, and $<_{lex}$, it is easy to verify the following result which we shall use continuously to make proofs by induction.

Lemma 3.6. *Let $\mathcal{F} \subseteq \mathbb{N}^{[<\infty]}$ and $M \in \mathbb{N}^{[\infty]}$. The following hold:*

- (i) *$\mathcal{F}_{\{n\}} \upharpoonright M = (\mathcal{F} \upharpoonright M)_{\{n\}}$, for $n \in M$,*
- (ii) *$\mathcal{F}_{\{n\}} = \bigcup_{m>n} (\mathcal{F}_{\{n\}})_{\{m\}} \oplus \{\{m\}\}$, for $n \in \mathbb{N}$,*
- (iii) *$\mathcal{F} \upharpoonright M = \bigcup_{n \in M} (\mathcal{F} \upharpoonright M)_{\{n\}} \oplus \{\{n\}\}$.*

Lemma 3.7. *Let $\mathcal{F}$ be an α-uniform family on a final segment S of $\mathbb{N}$, M an infinite subset of S, u a finite set and $0<\beta<\alpha$, then*
$$\left[(\mathcal{F}\restriction M)_u \oplus \{u\}\right]^{(\beta)} = \left[(\mathcal{F}\restriction M)_u\right]^{(\beta)} \oplus \{u\}.$$
In particular, for $n\in\mathbb{N}$ we have
$$\left[(\mathcal{F}\restriction M)_{\{n\}} \oplus \{\{n\}\}\right]^{(\beta)} = \left[(\mathcal{F}\restriction M)_{\{n\}}\right]^{(\beta)} \oplus \{\{n\}\}.$$
Proof. By induction on β. The result its true for $\beta=1$ by Lemma 3.2. Let us consider $\beta<\alpha$ and let us suppose that the lemma is true for all $\gamma<\beta$.

(i) Suppose $\beta=\gamma+1$ and let $t\in\left[(\mathcal{F}\restriction M)_u\oplus\{u\}\right]^{(\gamma+1)}$. Then there exists $(t_i)_i$ in $\left[(\mathcal{F}\restriction M)_u\oplus\{u\}\right]^{(\gamma)}$ such that $t_i\to t$. By the inductive hypothesis, $(t_i)_i\in\left[(\mathcal{F}\restriction M)_u\right]^{(\gamma)}\oplus\{u\}$. Thus applying Lemma 3.2 we get that $t/u\in\left[(\mathcal{F}\restriction M)_u\right]^{(\gamma+1)}$. Hence $t=u\cup t/u\in\left[(\mathcal{F}\restriction M)_u\right]^{(\gamma+1)}\oplus\{u\}$.

Reciprocally, let $t\in\left[(\mathcal{F}\restriction M)_u\right]^{(\gamma+1)}\oplus\{u\}$. Then $t/u\in\left[(\mathcal{F}\restriction M)_u\right]^{(\gamma+1)}$. Thus there is $(t_i)_i\in\left[(\mathcal{F}\restriction M)_u\right]^{(\gamma)}$ such that $t_i\to t/u$. Hence $t\in\left[(\mathcal{F}\restriction M)_{\{n\}}\oplus\{u\}\right]^{(\gamma+1)}$, because
$$u\cup t_i\in\left[(\mathcal{F}\restriction M)_u\right]^{(\gamma)}\oplus\{u\}=\left[(\mathcal{F}\restriction M)_u\oplus\{u\}\right]^{(\gamma)}.$$
(ii) If β is an ordinal limit, then
$$\begin{aligned}\left[(\mathcal{F}\restriction M)_u\oplus\{u\}\right]^{(\beta)} &= \bigcap_{\lambda<\beta}\left[(\mathcal{F}\restriction M)_u\oplus\{u\}\right]^{(\lambda)}\\ &= \bigcap_{\lambda<\beta}\left(\left[(\mathcal{F}\restriction M)_u\right]^{(\lambda)}\oplus\{u\}\right)\\ &= \left(\bigcap_{\lambda<\beta}\left[(\mathcal{F}\restriction M)_u\right]^{(\lambda)}\right)\oplus\{u\}\\ &= \left[(\mathcal{F}\restriction M)_u\right]^{(\beta)}\oplus\{u\}.\end{aligned}$$
□

Proposition 3.8. *Let $\mathcal{F}$ be an α-uniform family on a final segment of $\mathbb{N}$ with $2<\alpha<\omega_1$, $M\in\mathbb{N}^{[\infty]}$ and $0<\beta<\alpha$. If $t\in(\mathcal{F}\restriction M)^{(\beta)}$ then one of the following holds:*

(i) *$t/k\in((\mathcal{F}\restriction M)_{\{k\}})^{(\beta)}$, where $k=min(t)$, or*

(ii) *$t=t_p^{\mathcal{F}}$, for some $p\in\mathbb{N}$ with $p-1\in M$.*

Therefore
$$(\mathcal{F}\restriction M)^{(\beta)}\subseteq\bigcup_{k\in M}[(\mathcal{F}\restriction M)_{\{k\}}\oplus\{\{k\}\}]^{(\beta)}\cup\{t_p^{\mathcal{F}}:\ t_p^{\mathcal{F}}\subseteq M\ and\ p-1\in M\}.$$
Proof. Note that the last equation is consequence of (i), (ii) and Lemma 3.7. On the other hand, let $t\in(\mathcal{F}\restriction M)^{(\beta)}$ and $k=\min(t)$. Then $t_k^{\mathcal{F}}\leq_{lex} t<_{lex} t_{k+1}^{\mathcal{F}}$. There are two cases to consider: (a) Suppose $t=t_k^{\mathcal{F}}$. Since t is a limit point, then by lemma 3.4, $k-1\in M$ and (ii) holds.

(b) Suppose $t_k^{\mathcal{F}} <_{lex} t$. Let

$$U_k = \{s \in \mathcal{F}|M : \, t_k^{\mathcal{F}} <_{lex} s <_{lex} t_{k+1}^{\mathcal{F}}\}.$$

Then $t \in U_k$ and U_k is an open subset of $\mathcal{F} \upharpoonright M$. Thus $t \in (U_k)^{(\beta)} \subseteq ((\mathcal{F} \upharpoonright M)_{\{k\}} \oplus \{\{k\}\})^{(\beta)}$ $= ((\mathcal{F} \upharpoonright M)_{\{k\}})^{(\beta)} \oplus \{\{k\}\}$. Thus (i) holds.

□

3.1. **Finite CB derivative.** In this section we present some results about the finite derivatives $(\mathcal{F} \upharpoonright M)^{(l)}$, with $l < \omega$.

Lemma 3.9. *Let $\mathcal{F}$ be an α-uniform family on M with $\alpha \geq \omega$. There is a sequence $(w_j)_j$ of finite sets with $(min(w_j))_j$ increasing and an increasing sequence of integers $(k_j)_j$ such that $\mathcal{F}_{w_j}$ is k_j-uniform on M/w_j.*

Proof. By induction on α. For $\alpha = \omega$ the result follows from the definition of a ω-uniform family. If $\alpha > \omega$, then $\mathcal{F}_{\{j\}}$ is β_j-uniform on M/j with $\omega \leq \beta_j < \alpha$ for (eventually) all $j \in M$. Using the inductive hypothesis, define recursively k_j and v_j for $j \in M$ such that $\mathcal{F}_{\{j\} \cup v_j}$ is k_j-uniform on M/v_j, $j < min(v_j)$ and $(k_j)_j$ increasing. Take $w_j = \{j\} \cup v_j$ with $j \in M$.

□

Proposition 3.10. *Let $\mathcal{F}$ be an α-uniform family on a final segment S of $\mathbb{N}$ with $\alpha \geq 3$ and $M \in S^{[\infty]}$. Suppose there is $l \in \mathbb{N}$ with $1 \leq l$ and $N \in \mathbb{N}^{[\infty]}$ such that $\{i, i+1, i+2, \ldots, i+l\} \subseteq M$ for all $i \in N$. Let $u \in \mathbb{N}^{[<\infty]}$ and $p > max(u) + 1$ be such that $\mathcal{F}_{u \cup \{p-1\}}$ is β-uniform with $l \leq \beta$. If $t \in \mathcal{F}$ is of the form*

$$t = u \cup \{p, p+1, \ldots, p+m\}$$

with $l \leq m$, then $t \in (\mathcal{F} \upharpoonright M)^{(l)}$.

Proof. When $l = 1$, the result follows from 3.4, thus we assume $l \geq 2$. Let t, M and N as in the hypothesis. We will define a sequence $(s_i)_i$ in $(\mathcal{F} \upharpoonright M)^{(l-1)}$ converging to t.

We treat first the case $\beta < \omega$. When $l = \beta$, take $s_i = u \cup \{p-1\} \cup \{i+1, \cdots, i+l\}$ for $i \in N/p$. If $l < \beta$, then for infinite many $i \in N$ there is a nonempty finite set w_i such that

$$s_i = u \cup \{p-1\} \cup w_i \cup \{i+1, \cdots, i+l\} \in \mathcal{F} \upharpoonright M,$$

$p-1 < min(w_i)$, $max(w_i) < i$ and $min(w_i) \to \infty$. This finishes the definition of the sequence $(s_i)_i$. By a straightforward inductive argument, we conclude that $s_i \in (\mathcal{F} \upharpoonright M)^{(l-1)}$. By lemma 3.2, $s_i \to t$ and thus $t \in (\mathcal{F} \upharpoonright M)^{(l)}$.

Now suppose $\beta \geq \omega$. By lemma 3.9, there are sequences $(w_i)_i$ and $(k_i)_i$ such that $p < min(w_i) \to \infty$, $k_i > m$ and $\mathcal{F}_{u \cup \{p-1\} \cup w_i}$ is k_i-uniform. Then we construct the sequence $(s_i)_i$ as before.

□

For k-uniform families with $k \in \omega$ we have the following proposition.

Proposition 3.11. *Let $\mathcal{F}$ be a k-uniform family on a final segment of $\mathbb{N}$ with $3 \leq k$. Let $l \in \mathbb{N}$ with $2 \leq l < k$, $M \in \mathbb{N}^{[\infty]}$ and $t \subseteq M$. If $t \in (\mathcal{F} \upharpoonright M)^{(l)}$, then there exist $N \in \mathbb{N}^{[\infty]}$ such that $\{i, i+1, i+2, \ldots, i+l\} \subseteq M$ for all $i \in N$ and*

$$t = u \cup \{p, p+1, \ldots, p+m\}$$

for some $u \in \mathbb{N}^{[<\infty]}$ *with* $max(u) < p-1 \in M$ *and* $l \leq m \leq k-1$*.*

Proof. Let $t \in (\mathcal{F} \upharpoonright M)^{(l)}$, then by lemma 3.4

$$t = u \cup \{p, p+1, \ldots, p+m\}$$

for some $u \in \mathbb{N}^{[<\infty]}$ with $max(u) < p-1 \in M$. Let $(s_i)_i$ in $(\mathcal{F} \upharpoonright M)^{(l-1)}$ converging to t. By lemma 3.2 we assume that each s_i is of the form

$$s_i = u \cup \{p-1\} \cup v_i$$

with $p-1 < min(v_i)$.

The proof is by induction on l. By the inductive hypothesis when $l \geq 3$ and by lemma 3.4 when $l = 2$, we conclude that there is an increasing sequence $(p_i)_i$ such that $p_i - 1 \in M$, $\{p_i, p_i+1, \cdots, p_i+m_i\} \subseteq v_i$ and $l-1 \leq m_i$. In particular, this says that $\{p_i-1, p_i, p_i+1, \cdots, p_i+l-1\} \subset M$ for all i.

Now we show that $l \leq m < |t|-1$. In fact, $m = |t| - |u| - 1 = |v_i| \geq m_i + 1 \geq l$.

$\square$

From the previous results we immediately get the following:

Theorem 3.12. *Let* $M \in \mathbb{N}^{[\infty]}$ *and* $k > 2$*. Then* $M^{[k]}$*, as a subspace of* $\mathbb{N}^{[k]}$*, has CB index* k *if, and only if, there exists* $p \in \mathbb{N}$ *and* $N \in \mathbb{N}^{[\infty]}$ *such that* $\{p-1, p, p+1, p+2, \ldots, p+k-1\} \subseteq M$ *and* $\{i, i+1, i+2, \ldots, i+k-1\} \subseteq M$ *for all* $i \in N$*.* $\square$

The previous Theorem gives a characterization of those $M \in \mathbb{N}^{[\infty]}$ such that the CB index of $\mathcal{F} = \mathbb{N}^{[k]}$ and $\mathcal{F}|M$ are the same. However, this does not guarantee that $\mathcal{F}|M$ contains a topological copy of $\mathcal{F}$. To get this, we need that $\{p-1, p, p+1, p+2, \ldots, p+k-1\} \subseteq M$ for infinite many p.

The following example shows what we have said in the introduction about Nash-Williams theorem.

Example 3.13. *Let* $\mathcal{F}$ *be a* α*-uniform family on* $\mathbb{N}$ *with* $\alpha \geq 2$*. Let* $\mathcal{B} = \mathcal{F}^{(1)}$ *and* M *be an infinite set. We will show that* $(\mathcal{F} \upharpoonright M) \setminus \mathcal{B} \neq \emptyset$*. In particular, this says that if* M *is homogeneous for the partition given by* $\mathcal{B}$*, then* $(\mathcal{F} \upharpoonright M)$ *is a discrete subset of* $\mathcal{F}$*.*

Suppose first that $\alpha \geq \omega$*. By lemma 3.9, applied to* $\mathcal{F} \upharpoonright M$*, there is* $u \subset M$ *finite such that* $\mathcal{F}_u \upharpoonright M$ *is* k*-uniform for some* $2 \leq k < \omega$*. Let* $w \subset M$ *and* $p, q \in M$ *such that* $max(w) < p < q-1$ *and* $|w \cup \{p, q\}| = k$*. Then* $t = u \cup w \cup \{p, q\} \in \mathcal{F} \upharpoonright M$ *and* $t \notin \mathcal{B}$ *(by lemma 3.4). If* $\alpha < \omega$*, we can argue analogously to find* t*.*

4. $\mathcal{F}$-adequate sets

Let $\mathcal{F}$ be an α-uniform family on a final segment S of $\mathbb{N}$ with $\alpha \geq 2$. In this section we introduce the notion of a $\mathcal{F}$-adequate set M and later we will show that for those sets $\mathcal{F} \upharpoonright M$ has the same CB index as $\mathcal{F}$.

Let $M \in S^{[\infty]}$, we define by recursion a subset $M(\mathcal{F})$ of M and the notion of a $\mathcal{F}$-adequate set.

(*i*) If $\alpha = 2$, then $M(\mathcal{F})$ is the set of all $n \in M$ such that $t^{\mathcal{F}}_{n+1} \subset M$. And M is said to be $\mathcal{F}$-adequate, if $M(\mathcal{F})$ is not empty.

(*ii*) If $\alpha = \beta + 1$, then

$$M(\mathcal{F}) = \{n \in M : \ t^{\mathcal{F}}_{n+1} \subset M, M/n \text{ is } \mathcal{F}_{\{n\}}\text{-adequate and } (M/n)(\mathcal{F}_{\{n\}}) \text{ is infinite}\}.$$

And M is said to be $\mathcal{F}$-adequate, if $M(\mathcal{F})$ is not empty.

(*iii*) If α is limit, then $M(\mathcal{F}) = M$. Let $(\alpha_n)_n$ be the increasing sequence of ordinals as in the definition of a α-uniform family. We say that M is $\mathcal{F}$-adequate, if for all n there is a non empty finite set $v \subset M$ such that $\mathcal{F}_v$ is γ-uniform for some $\gamma \geq \alpha_n$ and M/v is $\mathcal{F}_v$-adequate.

Example 4.1. *If $\mathcal{F} = \mathbb{N}^{[2]}$, then an infinite set is $\mathcal{F}$-adequate when it contains three consecutive integers. In general, for $\mathcal{F} = \mathbb{N}^{[k+1]}$, a set is $\mathcal{F}$-adequate if it contains $\{n, n+1, \cdots, n+k\}$ for some n and infinite many intervals of length k.*

Let us say that an infinite set M is ω-adequate, if it contains arbitrarily long intervals of consecutive integers. Suppose $\mathcal{F}$ is ω-uniform on $\mathbb{N}$. Then M is $\mathcal{F}$-adequate iff M is ω-adequate.

Now suppose that $\mathcal{F}$ is $(\omega + 1)$-uniform on $\mathbb{N}$. Let P be a ω-adequate set. For a fixed $k \in \mathbb{N}$, let $M = P \cup \{k\} \cup t^{\mathcal{F}}_{k+1}$. Then M is $\mathcal{F}$-adequate. In fact, notice that $k \in M(\mathcal{F})$ because M/k is ω-adequate and $\mathcal{F}_{\{k\}}$ is ω-uniform.

The next lemma says that, in the definition of a $\mathcal{F}$-adequate set for α limit, we could have required that the ordinals γ are successor.

Lemma 4.2. *Let $\mathcal{F}$ be an α-uniform family on a final segment of $\mathbb{N}$ with α a limit ordinal. If M is an $\mathcal{F}$-adequate set, then there is a sequence of ordinals $\beta_n < \alpha$ and finite sets $u_n \subset M$ such that M/u_n is $\mathcal{F}_{u_n}$-adequate, $\mathcal{F}_{u_n}$ is $(\beta_n + 1)$-uniform on M/u_n, $\alpha = sup\{\beta_n : \ n \in \mathbb{N}\}$.*

Proof. By induction. The result holds for $\alpha = \omega$ by the definition of an ω-uniform family. Let $\alpha > \omega$ be a limit ordinal. Let $(\alpha_n)_n$ converging to α as in the definition of an α-uniform family. Fix sequences $(\gamma_n)_n$ and $(v_n)_n$ as in the definition of $\mathcal{F}$-adequate set. Since $(\alpha_n)_n$ is increasing, we assume that $\gamma_n > \alpha_n$. If there are infinitely many n such that γ_n is a successor ordinal, then we are done. Otherwise, assume that γ_n is a limit ordinal for all n. Apply the inductive hypothesis to $\mathcal{F}_{v_n}$ and M/v_n to get sequences of ordinals β^n_k converging to γ_n and finite sets $v^n_k \subset M$ such that $v_n \sqsubset v^n_k$, M/v^n_k is $\mathcal{F}_{v^n_k}$-adequate and $\mathcal{F}_{v^n_k}$ is $(\beta^n_k + 1)$-uniform. Now pick for each n an integer k_n such that $\beta^n_{k_n} > \alpha_n$. Take $u_n = v^n_{k_n}$ and $\beta_n = \beta^n_{k_n}$.

□

We going to present a method to construct $\mathcal{F}$-adequate sets. It is easy to show by induction on α that if $\mathcal{F}$ is α-uniform with $\alpha \geq \omega$, then there exist $s \in \mathbb{N}^{[<\infty]}$ such that $\mathcal{F}_s$ is ω-uniform on $\mathbb{N}/s$. Thus the following definition is non trivial.

Definition 4.3. *Let $\mathcal{F}$ be an α-uniform family with $\alpha \geq \omega$, we define the set $\mathcal{A}_{\mathcal{F}}$ as*

$$\mathcal{A}_{\mathcal{F}} = \{s \in \overline{\mathcal{F}}^{\sqsubseteq} : \mathcal{F}_s \textit{ is } \omega\textit{-uniform on } \mathbb{N}/s\}.$$

The set $\mathcal{A}_{\mathcal{F}}$ has the following properties:

(1) $\mathcal{A}_{\mathcal{F}}$ is infinite, if $\alpha \neq \omega$,
(2) $\mathcal{A}_{\mathcal{F}}$ is a front on M (If $\mathcal{F}$ is uniform on $M \in \mathbb{N}^{[\infty]}$),
(3) $\overline{\mathcal{A}_{\mathcal{F}}}^{\sqsubseteq}$ is a well founded tree.

From $\mathcal{A}_{\mathcal{F}}$ we define a $\mathcal{F}$-adequate tree and then a $\mathcal{F}$-adequate set of natural numbers.

Definition 4.4. *Let $\mathcal{F}$ be an α-uniform family with $\alpha \geq \omega$. We will say that a non empty subset T of $\overline{\mathcal{A}_{\mathcal{F}}}^{\sqsubseteq}$ is a **$\mathcal{F}$-tree**, if the following conditions hold*

(*i*) *If $t \in T$ and $s \sqsubseteq t$, then $s \in T$,*
(*ii*) *$Ter(T) \subseteq \mathcal{A}_{\mathcal{F}}$,*
(*iii*) *$\{n \in \mathbb{N} : n > t$ and $t \cup \{n\} \in T\}$ is infinite, for all $t \in T \backslash Ter(T)$,*

where $Ter(T)$ denotes the set of terminal nodes of T.

We remark that for an α-uniform family $\mathcal{F}$ on a set M with $\alpha > \omega$, $\mathcal{A}_{\mathcal{F}}$ is a front on M, and thus $\overline{\mathcal{A}_{\mathcal{F}}}^{\sqsubseteq}$ is well founded [1]. Thus each $\mathcal{F}$-tree is also well founded.

Definition 4.5. *Given $\mathcal{F}$ an α-uniform family with $\alpha > \omega$ and T a $\mathcal{F}$-tree, we define $E(T) \in \mathbb{N}^{[\infty]}$ as*

$$E(T) = \bigcup_{\substack{s \cup \{n\} \in T \\ s < n}} \{n\} \cup t_{n+1}^{\mathcal{F}_s}.$$

In other words,

$$\emptyset \neq \{x_0, x_1, x_2, \ldots x_{k-1}, x_k\} \in T \Leftrightarrow \{x_k\} \cup t_{x_k+1}^{\mathcal{F}_{\{x_0,x_1,x_2,\ldots x_{k-1}\}}} \subseteq E(T).$$

Lemma 4.6. *Let $\mathcal{F}$ be an α-uniform family over a final segment of $\mathbb{N}$ with $\alpha > \omega$ and $n \in \mathbb{N}$. Then,*

(1) *$(\mathcal{A}_{\mathcal{F}})_{\{n\}} = \mathcal{A}_{\mathcal{F}_{\{n\}}}$,*
(2) *If T is a $\mathcal{F}$-tree, then $T_{\{n\}}$ is a $\mathcal{F}_{\{n\}}$-tree for all n such that $\{n\} \in T$,*
(3) *$E(T_{\{n\}}) \subseteq E(T)$ for all n such that $\{n\} \in T$.*

Proof. It is straightforward. □

Proposition 4.7. *Let $\mathcal{F}$ be an α-uniform family over a final segment of $\mathbb{N}$ with $\alpha > \omega$. If T is a $\mathcal{F}$-tree, then $E(T)$ is $\mathcal{F}$-adequate.*

Proof. By induction on α. Let us fix a $\mathcal{F}$-tree T and let $M = E(T)$. We will show that M is $\mathcal{F}$-adequate and moreover that it is infinite.

(i) Suppose $\alpha = \omega + 1$. It is easy to verify that $n \in M$ for all n such that $\{n\} \in T$. Recall that by lemma 3.1, the size of $t_{n+1}^{\mathcal{F}}$ is increasing with n. Thus M contains arbitrarily long intervals of consecutive integers and by example 4.1, M is $\mathcal{F}_{\{n\}}$ adequate for all n.
(ii) If $\alpha = \beta + 1$, we will show that $M(\mathcal{F})$ contains all n such that $\{n\} \in T$. Fix such an n. Then $t_{n+1}^{\mathcal{F}} \subset M$. Let M_n be $E(T_{\{n\}})$. Since $T_{\{n\}}$ is a $\mathcal{F}_{\{n\}}$-tree, by the inductive hypothesis, M_n is $\mathcal{F}_{\{n\}}$-adequate and $M_n(\mathcal{F}_{\{n\}})$ is infinite. As $M_n(\mathcal{F}_{\{n\}}) \subset M_n \subset M/n$, then M/n is $\mathcal{F}_{\{n\}}$-adequate. Thus $n \in M(\mathcal{F})$.
(iii) Finally, suppose α is a limit ordinal. Then $T_{\{n\}}$ is a $\mathcal{F}_{\{n\}}$-tree for each n such that $\{n\} \in T$. Since $\mathcal{F}_{\{n\}}$ is α_n-uniform, then $E(T_{\{n\}})$ is $\mathcal{F}_{\{n\}}$-adequate. Since $E(T_{\{n\}}) \subseteq E(T)$, then $E(T)$ is also $\mathcal{F}_{\{n\}}$-adequate. As this holds for infinite many n's, then $E(T)$ is $\mathcal{F}$-adequate.

□

Example 4.8. *Let $\mathcal{F}$ be a $(\omega+1)$-uniform family on $\mathbb{N}$. It is easy to construct an infinite set P containing arbitrarily long intervals of consecutive natural numbers and such that $t_n^{\mathcal{F}} \not\subset P$ for all n. As in example 4.1, fixed $k \in \mathbb{N}$ and let $M = P \cup \{k\} \cup t_{k+1}^{\mathcal{F}}$. Then M is $\mathcal{F}$-adequate and it is not of the form $E(T)$ for any $\mathcal{F}$-tree T.*

5. Topological copies of $\mathcal{F}$ inside $\mathcal{F} \restriction M$

The following theorem is one of the main results of this paper. It justifies the introduction of $\mathcal{F}$-adequate sets.

Theorem 5.1. *Let $\mathcal{F}$ be an α-uniform family on a final segment S of $\mathbb{N}$ with $\alpha \geq 2$ and M a $\mathcal{F}$-adequate set. Then the CB index of $\mathcal{F} \restriction M$ is α.*

Proof. Since $\mathcal{F}$ is homeomorfic to ω^α, then the CB index of $\mathcal{F} \restriction M$ is at most α.

We first show by induction on $\beta \geq 1$ that if $\mathcal{F}$ is $(\beta+1)$-uniform, M is $\mathcal{F}$-adequate and $n \in M(\mathcal{F})$, then

$$t_{n+1}^{\mathcal{F}} \in (\mathcal{F} \restriction M)^{(\beta)}.$$

(*i*) If $\beta = 1$, then $t_{n+1}^{\mathcal{F}} = \{n+1, n+2\} \subset M$. From lemma 3.4, $t_{n+1}^{\mathcal{F}} \in (\mathcal{F} \restriction M)^{(1)}$.

(*ii*) Suppose $\beta = \gamma + 1$. Since M/n is $\mathcal{F}_{\{n\}}$-adequate and $(M/n)(\mathcal{F}_{\{n\}})$ is infinite, there is an increasing sequence $k_i \in (M/n)(\mathcal{F}_{\{n\}})$. Then by the inductive hypothesis, $t_{k_i+1}^{\mathcal{F}_{\{n\}}} \in (\mathcal{F}_{\{n\}} \restriction M)^{(\gamma)}$. By lemma 3.7 we have

$$s_i = \{n\} \cup t_{k_i+1}^{\mathcal{F}_{\{n\}}} \in (\mathcal{F} \restriction M)^{(\gamma)}.$$

By lemma 3.2, $s_i \to t_{n+1}^{\mathcal{F}}$. Thus $t_{n+1}^{\mathcal{F}} \in (\mathcal{F} \restriction M)^{(\gamma+1)}$ and we are done.

(*iii*) Suppose β is a limit ordinal. Let $\beta_m \uparrow \beta$ as in the definition of a β-uniform family. Since M/n is $\mathcal{F}_{\{n\}}$-adequate, then there is a sequence of finite sets $u_m \subset M/n$ and ordinals $\gamma_m \geq \beta_m$ such that $\mathcal{G}_m = \mathcal{F}_{\{n\} \cup u_m}$ is γ_m-uniform on M/u_m and M/u_m is $\mathcal{G}_m$-adequate. By lemma 4.2, we assume that each γ_n is a successor ordinal. Let $k_m \in M(\mathcal{G}_m)$. Then by the inductive hypothesis $t_{k_m+1}^{\mathcal{G}_m} \in (\mathcal{G}_m \restriction M)^{(\beta_m)}$. By lemma 3.7 we have

$$s_m = \{n\} \cup u_m \cup t_{k_m+1}^{\mathcal{G}_m} \in (\mathcal{F} \restriction M)^{(\beta_m)}.$$

By lemma 3.2, $s_m \to t_{n+1}^{\mathcal{F}}$. Thus $t_{n+1}^{\mathcal{F}} \in (\mathcal{F} \restriction M)^{(\beta)}$ and we are done.

The proof of the theorem is by induction on α. It remains only to consider the case when α is a limit ordinal. Let $(\alpha_k)_k$ be an increasing sequence of ordinals converging to α as in the definition of a α-uniform family. Since M is $\mathcal{F}$-adequate, then for all k there is a finite set $v_k \subset M$ such that M/v_k is $\mathcal{F}_{v_k}$-adequate and $\mathcal{F}_{v_k}$ is γ_k-uniform with $\gamma_k \geq \alpha_k$. By the inductive hypothesis, the CB index of $\mathcal{F}_{v_k} \restriction M/v_k$ is γ_k and therefore (by lemma 3.7) the CB index of $\mathcal{F} \restriction M$ is larger than γ_k for all k. Thus this last index is α.

□

For $\alpha = \omega$ we have a more precise result.

Theorem 5.2. *Let $\mathcal{F}$ be a ω-uniform family on a final segment of $\mathbb{N}$ and $M \in \mathbb{N}^{[\infty]}$. Then, $\mathcal{F} \restriction M$ has CB index ω if, and only if, M is $\mathcal{F}$-adequate.*

Proof. The *if* part follows from 5.1. For the other direction we will use the characterization of $\mathcal{F}$-adequate sets given in example 4.1.

Let $\mathcal{F}$ be a ω-uniform family on S and $(m_k)_k$ be an strictly increasing sequence in $\mathbb{N}$ such that $\mathcal{F}_{\{k\}}$ is m_k-uniform on S/k for all $k \in \mathbb{N}$. Suppose $\mathcal{F} \upharpoonright M$ has CB index ω. Then, given $n \in \mathbb{N}$ there exists $t \in \big(\mathcal{F} \upharpoonright M\big)^{(n)}$ and a sequence $(t_i)_i$ in $\big(\mathcal{F} \upharpoonright\ M\big)^{(n-1)}$ such that $t_i \uparrow t$. Let $k_i = min(t_i)$, by Proposition 3.8, for all $i \in \mathbb{N}$, $t_i/k_i \in \big((M/k_i)^{[m_{k_i}]}\big)^{(n-1)}$ or $t_i = t^{\mathcal{F}}_{k_i}$ with $k_i - 1 \in M$. Since $(t_i)_i$ is convergent, then eventually $t_i \neq t^{\mathcal{F}}_{k_i}$. Therefore, by Proposition 3.11, we can suppose that each t_i/k_i has the form $t_i/k_i = u_i \cup \{p_i, p_i+1, \ldots, p_i+n-1\}$ with $p_i - 1 \in M$ for each $i \in \mathbb{N}$. Hence, $\{p_i-1, p_i, p_i+1, \ldots, p_i+n-1\} \subseteq M$ for all $i \in \mathbb{N}$, which implies M is $\mathcal{F}$-adequate.

□

Corollary 5.3. *Let $\mathcal{F}$ be a ω-uniform family and $M \in \mathbb{N}^{[\infty]}$. Then, $\mathcal{F} \upharpoonright M$ has a topological copy of $\mathcal{F}$ if, and only if, M is $\mathcal{F}$-adequate.*

Proof. Let $\mathcal{F}$ be a ω-uniform family and $M \in \mathbb{N}^{[\infty]}$. If $\mathcal{F} \upharpoonright M$ contains a topological copy of $\mathcal{F}$, then $\mathcal{F} \upharpoonright M$ has CB index ω and therefore by Theorem 5.2 M is $\mathcal{F}$-adequate. Reciprocally, if M is an $\mathcal{F}$-adequate set, then by Theorem 5.2 $\mathcal{F} \upharpoonright M$ has CB index ω, and by Theorem 2.3 $\mathcal{F} \upharpoonright M$ has a topological copy of $\mathcal{F}$.

□

Finally, we present a result about the restriction to a set of the form $E(T)$ for T a $\mathcal{F}$-tree.

Theorem 5.4. *Let $\mathcal{F}$ be an α-uniform family with $\alpha > \omega$ indecomposable. If T is a $\mathcal{F}$-tree, then $\mathcal{F} \upharpoonright E(T)$ contains a topological copy of $\mathcal{F}$.*

Proof. Let $\mathcal{F}$, α and T be as in the hypothesis. Then by proposition 4.7, we know that $E(T)$ is $\mathcal{F}$-adequate. Hence by Theorem 5.1, $\mathcal{F} \upharpoonright E(T)$ has CB index α, and by Theorem 2.3, $\mathcal{F} \upharpoonright E(T)$ has a topological copy of $\mathcal{F}$. □

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